10.7: Are all the terms in these equations equally important? Let's use scale analysis.

Our equations of motion have several terms in them, and the question is, “Which ones are the largest and, thus, the most important?” The answer is, “It depends on the situation.” You can follow the following steps to determine which terms to keep and which you can safely ignore when you are trying to calculate the force balance for a specific atmospheric phenomenon. This process is called scale analysis and it can be applied to any conservation equation you would like to simplify.

1. Decide on the phenomenon of interest (e.g., cyclone, front, hurricane, tornado).
2. Determine the characteristic (i.e., typical) lengths and times over which the phenomenon occurs.
3. Determine the range of fluctuations of equation variables in space and time during the phenomenon.
4. Approximate derivatives (i.e., $\partial p/\partial x \sim \Delta p/\Delta x$)
5. Compare the magnitudes of the terms in the equation.
6. Keep only the relatively large terms (say, the top two orders of magnitude) and neglect the smaller terms.

The Coriolis Parameter

Define the Coriolis parameter as $f \equiv 2\Omega \sin \phi$. At $45^\circ$ N, $f = (2\times 7.27 \times 10^{-5} \text{s}^{-1}) \sin 45^\circ \sim 10^{-4} \text{s}^{-1}$.

We will use this parameter in the scale analysis in the next section and then throughout the rest of the lesson.

Example: Scale analysis of the averaged $x$-momentum equation for mid-latitude synoptic-scale flow in the free troposphere.
1. Phenomenon: mid-latitude synoptic-scale flow in the free troposphere.

2. \( L \sim 1000 \text{ km} = 10^6 \text{ m}; H \sim 10 \text{ km} = 10^4 \text{ m}; \) (in boundary layer only, \( C_d \sim 10^{-2} \) and \( h \sim 1000 \text{ m} \)) \( t \sim ? \)

3. \( U \sim 10 \text{ m s}^{-1}; \Delta p \sim 10 \text{ hPa} = 10^3 \text{ Pa}\) (in the horizontal)

\( t \sim L / u = \left( 10^6 \text{ m} \right) / \left( 10 \text{ m s}^{-1} \right) = 10^5 \text{ s} \sim 1 \text{ day} \)

\( f_0 \sim 2 \Omega \sin \phi_0 \sim 2 \Omega \cos \phi_0 \sim 10^{-4} \text{ s}^{-1} \)

\( W \sim H / t = 10^{-1} \text{ m s}^{-1}; a \sim 10^7 \text{ m}; \nu \sim 10^{-5} \text{ m}^2 \text{ s}^{-1}; \rho \sim 1 \text{ kg m}^{-3} \)

4 and 5.

<table>
<thead>
<tr>
<th>term</th>
<th>magnitude (variables)</th>
<th>magnitude (m s(^{-2}))</th>
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<tbody>
<tr>
<td>( \frac{D u}{D t} )</td>
<td>( \frac{U}{t} )</td>
<td>( \frac{10}{10^5} = 10^{-4} )</td>
</tr>
<tr>
<td>( 2 \Omega v \sin \phi )</td>
<td>( f_0 U )</td>
<td>( 10^{-4} \times 10 = 10^{-3} )</td>
</tr>
<tr>
<td>( -2 \Omega w \cos \phi )</td>
<td>( f_0 W )</td>
<td>( 10^{-4} \times 10^{-1} = 10^{-5} )</td>
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<td>( \frac{u w}{a} )</td>
<td>( \frac{U W}{a} )</td>
<td>( \frac{(10) \times \left( 10^{-1} \right)}{10^7} = 10^{-7} )</td>
</tr>
<tr>
<td>( \frac{u v \tan \phi}{a} )</td>
<td>( \frac{U^2}{a} )</td>
<td>( \frac{10^2}{10^7} = 10^{-5} )</td>
</tr>
<tr>
<td>(- \frac{1}{\rho} \frac{\partial p}{\partial x} )</td>
<td>( \frac{\Delta p}{\rho L} )</td>
<td>( \frac{10^3}{(1) \times \left( 10^6 \right)} = 10^{-3} )</td>
</tr>
<tr>
<td>(- \frac{C_d}{h}</td>
<td>\vec{V} \vec{V} )</td>
<td>( \frac{C_d U^2}{h} )</td>
</tr>
<tr>
<td>(- \nu \nabla^2 u )</td>
<td>( \frac{\nu U}{H^2} )</td>
<td>( \frac{\left( 10^{-5} \right) \times \left( 10 \right)}{10^8} = 10^{-12} )</td>
</tr>
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</table>

6. Using just the most important (i.e., biggest) terms and knowing that the turbulent drag term goes to zero above the atmospheric boundary layer, we can write a simplified \( x \)-momentum balance as:

\[ 0 = - \frac{1}{\rho} \frac{\partial p}{\partial x} + f v \]

Using the same scale analysis with the \( y \)-momentum equation, we can write a simplified \( y \)-momentum balance as:

\[ 0 = - \frac{1}{\rho} \frac{\partial p}{\partial y} - f u \]
Only two terms remain in both equations. One term is the pressure gradient force and the other is the Coriolis force. Since \( \frac{D u}{D t} \) is much smaller than either the pressure gradient force or the Coriolis force, these two forces must be about in balance. We call this balance the **Geostrophic Balance**. It is very important for understanding atmospheric dynamics and we will talk about its consequences in more detail later.

The molecular friction term is the smallest of all the terms for the case of large-scale flow in the atmosphere. This term is almost always very small for most meteorological phenomena, which is why we had eliminated it from the averaged momentum conservation equation earlier.

We ignored the acceleration term, \( Du/Dt \), because it is an order of magnitude smaller than the other two terms. We often must keep all terms that are within an order of magnitude of each other because our approximations may bias our results one way or the other. For instance, if we say that the velocity is 10 m s\(^{-1}\), the spatial scale is 100 km, and the pressure change is 10 hPa when more accurate numbers are more like 20 m s\(^{-1}\), 50 km, and 5 hPa, then we would be off almost an order of magnitude in our value for the centrifugal force, but get the same order of magnitude for the pressure gradient force. So, terms that are two orders of magnitude smaller than the rest you can easily neglect, but think carefully about terms that are only one order of magnitude different. For example, for very intense low-pressure systems, \( \frac{D u}{D t} \) must be considered because it can become about as large as the pressure gradient force and the Coriolis force.

When scale analysis is applied to the z-momentum equation for mid-latitude synoptic-scale flow, the result is the simplified z-momentum balance:

\[
0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g
\]

If you rearrange Equation 10.26, you will get the **hydrostatic equilibrium equation**, Equation 2.18. Previously we derived it using a balance of forces on a slab of air, but here it comes naturally out of the z-momentum equation.

The following video (4:19) provides further explanation on how to complete the above example:
Scale analysis is very important. Because it tells us which terms in any equation are the most important and which terms we can ignore. In scale analysis you do not need to know the exact values for the variables. But instead you need to only know their order of magnitude. The process is straightforward. First, determine the phenomenon of interest whether it be cyclone, front, hurricane, tornado, synoptic-scale, winter weather. Determine the characteristic-- that is typical lengths and times-- over which phenomenon occurs. Determine the range of fluctuations of equation variables in space and time during the phenomenon. Approximate derivatives, that is the partial of p with respect to x would become delta p over delta x where they're roughly estimated. Compare the magnitudes of terms in the equation. And then keep only the relatively large terms-- say the top two orders of magnitude-- and neglect the much smaller terms. Let's look at this example of the x momentum equation for mid-latitude synoptic-scale flow. So in this case it's mid-latitude synoptic-scale flow. The lake is about 1,000 kilometers, which is 10 and 6 meters. The height is about 10 kilometers, which is 10 to the 4 meters. And if we were in the boundary layer only, we would find that the friction drag coefficients is 10 to the minus 2. And the height of the boundary layer is about 1,000 meters. Now we know that u is about 10 meters a second, roughly. It could be a lot less and a lot more. But it's that order of magnitude. Delta p is about 10 millibar over the length of interest. We see that the time then is equal to the scale of the synoptic-scale flow divided by the velocity, which is 10 to the 6 divided by 10, or 10 to the 5 seconds which is about a day. And we see that the Coriolis parameter is about 10 to the minus 4 per second. And we can estimate other factors, such as the w velocity which is height divided by time. So that's about 10 to the minus 1 meters per second. And so on. We continue on looking at derivatives and other terms. And so, for instance, the acceleration in the u direction is about 10 meters per second divided by 10 to the 5, which is about 10 to the minus 4. And so that's the size of that term.
We see that the Coriolis term is about 10 to the minus 3. We see that other apparent terms are 10 to the minus 5 to 10 to the minus 7. They're quite a bit smaller. The pressure gradient force we see is 1 over the density, which is about 1 kilogram per meter cubed times the pressure difference which is about 10 to the 3 pascals divided by the distance, which is 10 to the 6 meters. So it's about 10 to minus 3. And we see that if we were in the boundary layer that the aerodynamic drag which causes friction is acting as friction. It's about 10 to the minus 3. So in the boundary layer we would need to consider this term because it's the same order of magnitude as the pressure gradient term and one of the larger terms. When we're not in the boundary layer then $c_{sub d}$ is actually very, very small. And this term is very small. We can ignore it. The last term is viscosity which is true friction. And we can see that for the case of viscosity is tiny. And therefore we can always ignore it for synoptic-scale flow. So when we look at the terms we have we see that we have away from the boundary layer we have two terms the count. That is we have the Coriolis term. And we have the pressure gradient. And those are the only two terms that we need to keep.